

The Itzykson-Zuber integral for $U(m|n)$

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Abstract

We compute the Itzykson-Zuber(IZ) integral for the superunitary group $U(m|n)$. As a consequence, we are able to find the non-zero correlations of superunitary matrices.

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1 Introduction

In recent times there has been an enormous amount of work devoted to the understanding of random surfaces and statistical systems on random surfaces. The range of application of these ideas include non-critical string theory as well as Quantum Chromodynamics (QCD) in the large N limit. Progress in this area has been possible because the mathematical knowledge on random matrices has increased dramatically in the last fifteen years [1].

An important mathematical object that appears naturally in the discussion of random matrices is the integral over the unitary group [2]. This integral has been applied to the solution of the Two matrix model [2], [3] and, more recently, to the Migdal-Kazakov model of "induced QCD" [4].

On the other hand, there is considerable expectation that supersymmetry might play an important role in the physical world [5]. Since there is a natural extension of matrices to supermatrices [6], it is important to understand the properties of random supermatrices and study their relevance for the theory of random surfaces.

In this paper we will start the study of random supermatrices related to a supersymmetric extension of the Itzykson-Zuber integral. In particular we will calculate some non-vanishing correlations of superunitary matrices. The paper is organized as follows : section 2 contains a very brief review of some basic properties of Grassmannian manifolds, which will be used in the sequel. Section 3 contains the main result of this paper, which is the calculation of the IZ integral for $U(m|n)$. This result is subsequently used to find some particular correlations among the supermatrices elements.

2 Basic properties of a superspace

Since the purpose of this paper is to extend the Itzykson-Zuber integral to the case of the supergroup $U(m|n)$, which can be described by matrices acting on a superspace (supermatrices), we briefly review some of the basic properties of the linear algebra together with the differential and integral calculus defined over a Grassmann algebra. This sets the stage for the next section and also fixes our notation. For a more detailed and complete reference on these matters the reader is referred to Ref.[6].

Let us consider a superspace with coordinates $z^P = (q^i, \theta^\alpha)$, $i = 1, \dots, m$, $\alpha = 1, \dots, n$ such that q^i (θ^α) are even (odd) elements of a Grassmann algebra. This means that $z^P z^Q = (-1)^{\epsilon(P)+\epsilon(Q)} z^Q z^P$, where $\epsilon(P)$ is the Grassmann parity of the index P defined by $\epsilon(i) = 0, \text{ mod}(2)$; $\epsilon(\alpha) = 1, \text{ mod}(2)$. Also we have that $\epsilon(z^{P_1} z^{P_2} \dots z^{P_k}) = \sum \epsilon(P_i)$. The above multiplication rule implies in particular that any odd element of the Grassmann algebra has zero square, i.e. it is nilpotent.

Supermatrices are arrays that act linearly on the supercoordinates leaving invariant the partition among even and odd coordinates. To be more specific, the supercoordinates can be thought as forming an $(m+n) \times 1$ column vector with the first m entries (last n entries) being even (odd) elements of the Grassman algebra. In this way, an $(m+n) \times (m+n)$ supermatrix is an array written in the partitioned block form

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad (1)$$

where the constituent matrices have the entries A_{ij} , $B_{i\alpha}$, $C_{\alpha i}$, $D_{\alpha\beta}$. Besides, A_{ij} , $D_{\alpha\beta}$ ($B_{i\alpha}$, $C_{\alpha i}$) are even (odd) elements of the Grassmann algebra in such a way that the parity array of the supercoordinate vector column is

preserved. The parity of any supermatrix element is $\epsilon(M_{PQ}) = \epsilon(P) + \epsilon(Q)$ and defines the multiplication among supermatrix elements. The addition and multiplication of supermatrices according to the rules

$$(M_1 + M_2)_{PQ} = (M_1)_{PQ} + (M_2)_{PQ}, \quad (M_1 M_2)_{PQ} = (M_1)_{PR}(M_2)_{RQ},$$

is such that it produces again a supermatrix. The inverse of a supermatrix can be constructed in block form, in complete analogy with the classical case and it is well defined provided A^{-1} and D^{-1} exist. The inverse of an even matrix is calculated in the standard way.

The basic invariant of a supermatrix under similarity transformations is the supertrace

$$Str(M) = Tr(A) - Tr(D) = \sum_{P=1}^{m+n} (-1)^{\epsilon(P)} M_{PP},$$

which is defined so that the cyclic property $Str(M_1 M_2) = Str(M_2 M_1)$ is fulfilled for arbitrary supermatrices M_1, M_2 . The above definition of the supertrace leads to the construction of the superdeterminant in the form $Sdet(M) = \exp[Str(\ln M)]$, which is explicitly given in the following two equivalent forms [7]

$$Sdet(M) = \frac{\det(A - BD^{-1}C)}{\det(D)} = \frac{\det(A)}{\det(D - CA^{-1}B)}. \quad (2)$$

The above expression is written only in terms of even matrices in such a way that the determinant has its usual meaning. The superdeterminant has the multiplicative property $Sdet(M_1 M_2) = Sdet(M_1) Sdet(M_2)$.

The definition of the adjoint supermatrix follows the usual steps by requiring the identity $(y^{P*} M_{PQ} z^Q)^* = z^{P*} M^\dagger_{PQ} y^Q$, for an arbitrary bilinear form in the complex supercoordinates y^P , where $*$ denotes complex conjugation. Since the usual definition of complex conjugation in a Grassmann

algebra $(y^P y^Q)^* = y^{Q*} y^{P*}$ reverses the order of the factors without introducing any sign factor, we have the result $M^\dagger_{PQ} = M_{QP}^*$ as in the standard case.

A hermitian $(m+n) \times (m+n)$ supermatrix M is such that $M^\dagger = M$ and it has $(m+n)^2$ real independent components. The following properties are also fulfilled: (i) $(M^\dagger)^\dagger = M$, (ii) $(M_1 M_2)^\dagger = M_2^\dagger M_1^\dagger$ and (iii) $Sdet(M^\dagger) = Sdet(M)^*$. A unitary $(m+n) \times (m+n)$ supermatrix U is such that $UU^\dagger = U^\dagger U = I$ (where I is the identity supermatrix) and also has $(m+n)^2$ real independent components, which have the additional property that $(Sdet U)(Sdet U)^* = 1$. The set of all $(m+n) \times (m+n)$ unitary supermatrices form a group, called the supergroup $U(m|n)$, under the operation of supermatrix multiplication.

In the next section we will deal with derivation and integration with respect to the elements of a supermatrix, which will be considered as supercoordinates there. To this end, we summarize here the basic properties involved, in terms of the supercoordinates z^A . Differentiation and integration over the even supercoordinates follow the same rules and properties as the corresponding operations with complex numbers. On the other hand, the situation with respect to the odd supercoordinates is greatly simplified because of the nilpotency property, which leads to the conclusion that an arbitrary superfunction $F(q^i, \theta^\alpha)$ can be expanded into a finite set of products of the odd supercoordinates. In fact we have

$$F(q^i, \theta^\alpha) = f(q^i) + f_{\alpha_1}(q^i)\theta^{\alpha_1} + f_{\alpha_1\alpha_2}(q^i)\theta^{\alpha_1}\theta^{\alpha_2} + \dots + f_{\alpha_1\dots\alpha_n}(q^i)\theta^{\alpha_1}\dots\theta^{\alpha_n}, \quad (3)$$

where all the functions $f_{\alpha_1\dots\alpha_k}$ are completely antisymmetric in all the subindices. Also, a particular odd supercoordinate θ^{α_k} can only appear linearly in the above expansion. Right and left derivatives with respect to the odd coordi-

nates are defined according to

$$dF(q^i, \theta_\alpha) = \frac{\partial^R F}{\partial \theta^\alpha} d\theta^\alpha = d\theta^\alpha \frac{\partial^L F}{\partial \theta^\alpha}, \quad (4)$$

where the operator d has zero Grassman parity and acts distributively upon a product of odd coordinates: $d(\theta^{\alpha_1} \theta^{\alpha_2} \dots \theta^{\alpha_k}) = d\theta^{\alpha_1} \theta^{\alpha_2} \dots \theta^{\alpha_k} + \theta^{\alpha_1} d\theta^{\alpha_2} \dots \theta^{\alpha_k} + \dots + \theta^{\alpha_1} \theta^{\alpha_2} \dots d\theta^{\alpha_k}$. Both, right and left derivatives satisfy adequate Leibnitz rules which can be directly obtained from the definitions (4). Clearly, for even coordinates right and left derivatives coincide.

Integration over odd Grassmann variables is defined, according to Berezin, by the basic properties [8]

$$\int d\theta^\alpha = 0, \quad \int d\theta^\alpha \theta^\beta = \delta^{\alpha\beta}, \quad (5)$$

which allows the calculation of integrals over arbitrary functions using the expansion (3) together with the multiplication rules of supernumbers. The full integration measure of the superspace under consideration is given by $[dz] = dq^1 \dots dq^m d\theta^1 \dots d\theta^n$, where a particular order in the odd differentials has been chosen. Under a change of supercoordinates $z'^P = z'^P(z^Q)$ the integration measure transforms as

$$[dz'] = S \det\left(\frac{\partial^R z'^P}{\partial z^Q}\right) [dz]. \quad (6)$$

3 Integration over the unitary supergroup $U(m|n)$

In this section we calculate the extension to $(m+n) \times (m+n)$ supermatrices of the IZ integral. Our method, mutatis mutandis, follows closely to that of Itzykson and Zuber [2]. Let us consider the integral

$$\tilde{I}(M_1, M_2; \beta) \equiv \int [dU] e^{\beta \text{Str}(M_1 U M_2 U^\dagger)}, \quad (7)$$

where M_1, M_2 are hermitian supermatrices which can be diagonalized [9] and β is an even parameter. Up to a normalization factor μ (to be fixed later), we define the integration measure over $U(m|n)$ by

$$[dU] = \mu \prod_{P,Q=1}^{m+n} dU_{PQ} dU_{PQ}^* \delta(UU^\dagger - I). \quad (8)$$

Here the δ -function really means the product of $(m+n)^2$ unidimensional δ -functions corresponding to the independent constraints set by the condition $UU^\dagger = I$. In the case of an odd Grassmann variable θ ,

$$\delta(\theta - \bar{\theta}) = (\theta - \bar{\theta}) = \int d\pi e^{\pi(\theta - \bar{\theta})},$$

where π is another odd integration variable. It is important to observe that the measure in (7) possesses $2mn$ real independent odd differentials.

The measure (8) is invariant under independent right and left multiplication by arbitrary unitary supermatrices. In this way, \tilde{I} depends only upon the eigenvalues of M_1 and M_2 , which are given in the corresponding diagonal supermatrices Λ_1 and Λ_2 . Our notation is such that the first m eigenvalues of Λ are identified by λ_i , while the remaining n eigenvalues are denoted by $\bar{\lambda}_\alpha$. Such partition is characterized by the following parity assignment of the eigenvector components $V_P, \bar{V}_P : \epsilon(V_P) = \epsilon(P), \epsilon(\bar{V}_P) = \epsilon(P) + 1$. Before sketching the calculation of the basic integral given in Eq. (7), we state our final result and make some general comments. We obtain

$$\begin{aligned} \tilde{I}(\Lambda_1, \Lambda_2; \beta) = & \Sigma(\lambda_1, \bar{\lambda}_1) \Sigma(\lambda_2, \bar{\lambda}_2) \beta^{mn} \times (\beta)^{-\frac{m(m-1)}{2}} (-\beta)^{-\frac{n(n-1)}{2}} \times \\ & \times \prod_{p=1}^{m-1} p! \prod_{q=1}^{n-1} q! \frac{\det(e^{\beta \lambda_{1i} \lambda_{2j}})}{\Delta(\lambda_1) \Delta(\lambda_2)} \frac{\det(e^{-\beta \bar{\lambda}_{1\alpha} \bar{\lambda}_{2\beta}})}{\Delta(\bar{\lambda}_1) \Delta(\bar{\lambda}_2)}. \end{aligned} \quad (9)$$

Here, Δ is the usual Vandermonde determinant

$$\Delta(\lambda) = \prod_{i>j} (\lambda_i - \lambda_j), \quad \Delta(\bar{\lambda}) = \prod_{\alpha>\beta} (\bar{\lambda}_\alpha - \bar{\lambda}_\beta) \quad (10)$$

and the new function that appears is

$$\Sigma(\lambda, \bar{\lambda}) = \prod_{i=1}^m \prod_{\alpha=1}^n (\lambda_i - \bar{\lambda}_\alpha). \quad (11)$$

We observe that the polynomial $\Sigma(\lambda, \bar{\lambda})$ is completely symmetric under independent permutations of $\lambda, \bar{\lambda}$.

The expression (9) is completely determined up to a normalization factor related to that of the measure in equation (8). This situation is analogous to the standard IZ case where the required factor can be fixed directly from the corresponding expression by taking the limit $\Lambda_1, \Lambda_2 \rightarrow 0$ in a convenient way and demanding $\int [dU] = 1$, for example. This procedure leads to the correct factors in Eq.(3.4) of Ref.[2]. In our case, a similar limiting procedure leads to the conclusion that $\int [dU] \equiv 0$, precisely due to the appearance of the $\Sigma(\lambda, \bar{\lambda})$ functions in the numerator. This is not an unexpected result since we are dealing with odd Grassmann numbers. For this reason we have chosen the normalization factor in such a way that

$$\tilde{I}(\Lambda_1, \Lambda_2; \beta) = \Sigma(\lambda_1, \bar{\lambda}_1) \Sigma(\lambda_2, \bar{\lambda}_2) \beta^{mn} I(\lambda_1, \lambda_2; \beta) I(\bar{\lambda}_1, \bar{\lambda}_2; -\beta),$$

where $I(d_1, d_2; \beta)$ is the corresponding IZ integral.

Now we give some details of the proof of our result (9). We begin by constructing the superunitary invariant Laplacian operator on hermitian supermatrices, which is given by

$$\tilde{D} \equiv \sum_{P, Q=1}^{m+n} (-1)^{\epsilon(P)} \frac{\partial^2}{\partial M_{PQ} \partial M_{QP}}, \quad (12)$$

where the derivatives are both either left or right derivatives. This is a particular case of the superlaplacian constructed for curvilinear coordinates and corresponds to the metric $ds^2 = \text{Str}(dM^2)$ which is invariant under

global unitary transformations $dM \rightarrow U dM U^\dagger$ for hermitian supermatrices M .

Next we consider the propagator

$$\tilde{f}(M_1, M_2; t) \equiv \langle M_1 | e^{\frac{it}{2} \tilde{D}_1} | M_2 \rangle, \quad (13)$$

which, by construction, satisfies the differential equation

$$\left(\frac{\partial}{\partial t} - \frac{i}{2} \tilde{D}_1 \right) \tilde{f}(M_1, M_2; t) = 0, \quad (14)$$

together with the boundary condition

$$\tilde{f}(M_1, M_2; 0) = \langle M_1 | M_2 \rangle = \delta(M_1 - M_2), \quad (15)$$

at $t=0$. It can be shown that the propagator \tilde{f} is given by

$$\tilde{f}(M_1, M_2; t) = \frac{i^{n^2}}{(2\pi)^{mn}} \frac{1}{(2\pi it)^{(m-n)^2/2}} e^{\frac{i}{2t} \text{Str}(M_1 - M_2)^2}, \quad (16)$$

in analogy with the standard case. Since the supertrace of a squared supermatrix is not positive definite, the factor i of the exponential in (13) is included to insure convergence in further manipulations.

Let us now consider the time evolution of a wave function $\tilde{g}(M_1; t)$ satisfying (14) and such that at $t = 0$ reduces to a known function $\tilde{g}(M_1; 0) = \tilde{g}(M_1)$, which is invariant under $U(m|n)$ transformations. As a consequence, \tilde{g} is a function of the eigenvalues λ 's of M only and it is symmetric under the separate permutations of the sets $\{\lambda_i\}$ and $\{\bar{\lambda}_\alpha\}$. Such a wave function can be constructed using the propagator in Eq.(16), as

$$\tilde{g}(M_1; t) = \int [dM] \tilde{f}(M_1, M; t) \tilde{g}(M; 0), \quad (17)$$

where the integration measure over hermitian supermatrices is given by $[dM] = \prod_P dM_{PP} \prod_{S,R>S} dM_{RS} dM_{RS}^*$ and it is invariant under $U(m|n)$ transformations.

Now, we make a change of integration variables, from the initial M 's to radial (Λ) and angular (U) variables given by $M = U\Lambda U^\dagger$. This change of variables leads to

$$\tilde{g}(M_1; t) = \int [d\Lambda][dU] J \tilde{f}(M_1, U\Lambda U^\dagger; t) \tilde{g}(\Lambda; 0), \quad (18)$$

where the jacobian J is to be determined. The jacobian factorizes in the form $J = \tilde{\Delta}^2(\Lambda) J_1(U)$. The piece $J_1(U)$ will remain incorporated to $[dU]$ while we will find $\tilde{\Delta}^2$ explicitly. Let us observe that according to Ref.[9] there are some restrictions to diagonalize a hermitian supermatrix. We are assuming that such forbidden points constitute a set of zero measure in the configuration space defined by $\{M_{PQ}\}$. In virtue of the invariance of \tilde{f} under $U(m|n)$, with the particular choice arising from $M_1 = V\Lambda_1 V^\dagger$, together with the invariance of $[dU]$ under right and left multiplications, we arrive at the result

$$\tilde{g}(\Lambda_1; t) = \int [d\Lambda][dU] \tilde{\Delta}^2(\Lambda) \tilde{f}(\Lambda_1, U\Lambda U^\dagger; t) \tilde{g}(\Lambda), \quad (19)$$

which shows that the wave function preserves its invariance under $U(m|n)$ transformations.

The next step is to construct the wave function

$$\tilde{\xi}(\Lambda; t) \equiv \tilde{\Delta}(\Lambda) \tilde{g}(\Lambda; t) \quad (20)$$

and to realize that

$$\tilde{K}(\Lambda_1, \Lambda; t) = \tilde{\Delta}(\Lambda_1) \tilde{\Delta}(\Lambda) \int [dU] \tilde{f}(\Lambda_1, U\Lambda U^\dagger; t), \quad (21)$$

is the kernel that corresponds to the propagator of $\tilde{\xi}(\Lambda; t)$.

Before calculating such kernel in an independent way, so that we can use Eq.(21) to find the value for the angular integration, we now proceed to the

calculation of the jacobian $\tilde{\Delta}^2$. This calculation is equivalent to finding the volume element in spherical coordinates. The general property to be used here originates in the differential geometry of supermanifolds [6] and states that given a metric tensor such that the length invariant is $ds^2 = dz^P g_{PQ} dz^Q$, the correct integration measure over the manifold is given by $\sqrt{g} [dz]$, where $g = Sdet(g_{PQ})$. As we mentioned previously, the adequate length element in the case of hermitian supermatrices is $ds^2 = Str(dM^2)$. After making the change of variables $M = U\Lambda U^\dagger$ we are left with

$$ds^2 = Str(d\Lambda^2) + Str([U^\dagger dU, \Lambda]^2). \quad (22)$$

Since we are only interested in extracting the Λ dependent piece of $\sqrt{g} = \sqrt{g_\Lambda g_U}$ we can further consider the representation $U = e^{iH}$, in terms of another hermitian supermatrix H . Taking H to be infinitesimal in such a way that $U^\dagger dU = idH$, we obtain

$$\begin{aligned} ds^2 = & \sum_{i,j=1}^m \delta_{ij} d\lambda_i d\lambda_j - \sum_{\alpha,\beta=1}^n \delta_{\alpha\beta} d\bar{\lambda}_\alpha d\bar{\lambda}_\beta + \\ & + \sum_{a,b=1}^m (\lambda_a - \lambda_b)^2 dH_{ab} dH_{ba} + \sum_{\alpha,\beta=1}^n (\bar{\lambda}_\alpha - \bar{\lambda}_\beta)^2 dH_{\alpha\beta} dH_{\beta\alpha} - \\ & - \sum_{a,\beta=1}^{n,m} (\lambda_a - \bar{\lambda}_\beta)^2 dH_{a\beta} dH_{\beta a} - \sum_{\alpha,b=1}^{m,n} (\bar{\lambda}_\alpha - \lambda_b)^2 dH_{\alpha b} dH_{b\alpha}, \end{aligned} \quad (23)$$

for the expression (22). From here we conclude that

$$\tilde{\Delta}(\Lambda) = \frac{\prod_{i>j}(\lambda_i - \lambda_j) \prod_{\alpha>\beta}(\bar{\lambda}_\alpha - \bar{\lambda}_\beta)}{\prod_{i,\alpha}(\lambda_i - \bar{\lambda}_\alpha)} = \frac{\Delta(\lambda)\Delta(\bar{\lambda})}{\Sigma(\lambda, \bar{\lambda})}. \quad (24)$$

Having completely determined the wave function $\tilde{\xi}(\Lambda; t)$ of eq.(20), we now calculate its equation of motion. The starting point is eq.(14) for the function $\tilde{g}(\Lambda; t) = \tilde{\xi}/\tilde{\Delta}$. Also we will need the expression of the superlaplacian

in curvilinear coordinates, which is given by

$$\tilde{D}\Psi = \frac{1}{\sqrt{g}} \sum_{P,Q=1}^{m+n} (-1)^{\epsilon_P} \frac{\partial^L}{\partial z^P} (g^{PQ} \sqrt{g} \frac{\partial^L \Psi}{\partial z^Q}), \quad (25)$$

where g^{PQ} is the inverse of the metric tensor g_{PQ} . The expression (25) further simplifies in our case, since we are applying the superlaplacian to a function which does not depend on the angular variables. In this way we obtain

$$\frac{\partial \tilde{\xi}(\Lambda; t)}{\partial t} = \frac{i}{2\tilde{\Delta}} \left[\sum_{i=1}^m \frac{\partial}{\partial \lambda_i} (\tilde{\Delta}^2 \frac{\partial}{\partial \lambda_i} (\frac{\tilde{\xi}}{\tilde{\Delta}})) - \sum_{\alpha=1}^n \frac{\partial}{\partial \bar{\lambda}_\alpha} (\tilde{\Delta}^2 \frac{\partial}{\partial \bar{\lambda}_\alpha} (\frac{\tilde{\xi}}{\tilde{\Delta}})) \right], \quad (26)$$

for the time evolution of $\tilde{\xi}(\Lambda; t)$, where we have dropped the reference to left derivatives since all variables are even. The above equation reduces to

$$\frac{\partial \tilde{\xi}(\Lambda; t)}{\partial t} = \frac{i}{2} \left[\sum_{i=1}^m \frac{\partial^2}{\partial \lambda_i^2} - \sum_{\alpha=1}^n \frac{\partial^2}{\partial \bar{\lambda}_\alpha^2} \right] \tilde{\xi}(\Lambda; t), \quad (27)$$

in virtue of the following property of the function $\tilde{\Delta}$

$$\left(\sum_{i=1}^m \frac{\partial^2}{\partial \lambda_i^2} - \sum_{\alpha=1}^n \frac{\partial^2}{\partial \bar{\lambda}_\alpha^2} \right) \tilde{\Delta} \equiv 0. \quad (28)$$

The above equation can be proved from the expression

$$\begin{aligned} \sum_{i=1}^m \frac{\partial^2 \tilde{\Delta}}{\partial \lambda_i^2} = \tilde{\Delta} [& -2 \sum_{\alpha, i, j \neq i} \frac{1}{(\lambda_i - \bar{\lambda}_\alpha)(\lambda_i - \lambda_j)} + 2 \sum_{i, \alpha} \frac{1}{(\lambda_i - \bar{\lambda}_\alpha)^2} + \\ & + \sum_{i, \alpha, \beta \neq \alpha} \frac{1}{(\lambda_i - \bar{\lambda}_\alpha)(\lambda_i - \bar{\lambda}_\beta)}], \end{aligned} \quad (29)$$

together with the analogous relation

$$\begin{aligned} \sum_{\alpha=1}^n \frac{\partial^2 \tilde{\Delta}}{\partial \bar{\lambda}_\alpha^2} = \tilde{\Delta} [& 2 \sum_{\alpha, i} \frac{1}{(\lambda_i - \bar{\lambda}_\alpha)^2} + \sum_{\alpha, i, j \neq i} \frac{1}{(\lambda_i - \bar{\lambda}_\alpha)(\lambda_j - \bar{\lambda}_\alpha)} + \\ & + 2 \sum_{\alpha, \beta \neq \alpha, i} \frac{1}{(\lambda_i - \bar{\lambda}_\alpha)(\bar{\lambda}_\alpha - \bar{\lambda}_\beta)}]. \end{aligned} \quad (30)$$

The calculation of the above equations (29) and (30) has already used the identity

$$\frac{1}{(\mu_1 - \mu_2)(\mu_1 - \mu_3)} + \frac{1}{(\mu_2 - \mu_3)(\mu_1 - \mu_2)} + \frac{1}{(\mu_3 - \mu_1)(\mu_3 - \mu_2)} = 0,$$

for the cases $\mu = \lambda, \bar{\lambda}$. The difference in the LHS of Eq.(28) further reduces to

$$\left(\sum_{i=1}^m \frac{\partial^2}{\partial \lambda_i^2} - \sum_{\alpha=1}^n \frac{\partial^2}{\partial \bar{\lambda}_\alpha^2} \right) \tilde{\Delta} = \tilde{\Delta} \left[\sum_i \sum_{\alpha, \beta \neq \alpha} A_{\alpha\beta i} - \sum_\alpha \sum_{i, j \neq i} B_{ij\alpha} \right], \quad (31)$$

with

$$A_{\alpha\beta i} = \frac{\bar{\lambda}_\alpha + \bar{\lambda}_\beta - 2\lambda_i}{(\lambda_i - \bar{\lambda}_\alpha)(\lambda_i - \bar{\lambda}_\beta)(\bar{\lambda}_\alpha - \bar{\lambda}_\beta)}, \quad (32)$$

$$B_{ij\alpha} = \frac{\lambda_i + \lambda_j - 2\bar{\lambda}_\alpha}{(\lambda_i - \bar{\lambda}_\alpha)(\lambda_i - \lambda_j)(\lambda_j - \bar{\lambda}_\alpha)}. \quad (33)$$

The result given in Eq.(28) is finally obtained due to the antisymmetry properties $A_{\alpha\beta i} = -A_{\beta\alpha i}$, $B_{ij\alpha} = -B_{ji\alpha}$.

The last step in the proof of Eq.(9) is the observation that the kernel $K(\Lambda_1, \Lambda_2; t)$ in eq.(21) satisfies the following differential equation

$$\frac{\partial K(\Lambda_1, \Lambda_2; t)}{\partial t} = \frac{i}{2} \left(\sum_{i=1}^m \frac{\partial^2}{\partial \lambda_{1i}^2} - \sum_{\alpha=1}^n \frac{\partial^2}{\partial \bar{\lambda}_{1\alpha}^2} \right) K(\Lambda_1, \Lambda_2; t), \quad (34)$$

with the initial condition

$$K(\Lambda_1, \Lambda_2; 0) = \delta(\Lambda_1 - \Lambda_2). \quad (35)$$

Moreover we require that $K(\Lambda_1, \Lambda_2; t)$ be separately antisymmetric under permutations of the $\{\lambda_i\}$ and the $\{\bar{\lambda}_\alpha\}$. Then, the solution of Eq. (34) is

$$K(\Lambda_1, \Lambda_2; t) = \frac{1}{(2\pi i t)^{(m+n)/2}} \frac{1}{m!} \frac{1}{n!} \det(e^{i(\lambda_{1,i} - \lambda_{2,j})^2/2t}) \det(e^{-i(\bar{\lambda}_{1,\alpha} - \bar{\lambda}_{2,\beta})^2/2t}), \quad (36)$$

which has the same symmetry properties of $\tilde{\xi}(\Lambda; t)$. From Eqs.(16), (21) and (36) we obtain

$$\int dU e^{-\frac{i}{t} \text{Str}(\Lambda_1 U \Lambda_2 U^\dagger)} = \text{factor} \times \left(\frac{1}{it}\right)^{mn} \left(\frac{1}{it}\right)^{-\frac{m(m-1)}{2}} \left(-\frac{1}{it}\right)^{-\frac{n(n-1)}{2}} \\ \times e^{-\frac{i}{2t} \text{Str}(\Lambda_1^2 + \Lambda_2^2)} \frac{\det(e^{i(\lambda_{1,i} - \lambda_{2,j})^2/2t}) \det(e^{(-i\bar{\lambda}_{1,\alpha} - \bar{\lambda}_{2,\beta})^2/2t})}{\tilde{\Delta}(\lambda_1) \tilde{\Delta}(\lambda_2)}, \quad (37)$$

where *factor* is independent of Λ_1 and Λ_2 . Now, we use that $e^{\text{Str}(\ln(A))} = S\det(A)$ with $A = e^{-\frac{i}{2t}(\Lambda_1^2 + \Lambda_2^2)}$ to get

$$\int [dU] e^{\beta \text{Str}(\Lambda_1 U \Lambda_2 U^\dagger)} = \text{factor} \times \beta^{mn} \left[(\beta)^{-\frac{m(m-1)}{2}} (-\beta)^{-\frac{n(n-1)}{2}} \right] \\ \times \frac{\det(e^{\beta \lambda_{1,i} \lambda_{2,j}}) \det(e^{-\beta \bar{\lambda}_{1,\alpha} \bar{\lambda}_{2,\beta}})}{\tilde{\Delta}(\lambda_1) \tilde{\Delta}(\lambda_2)}, \quad (38)$$

where $\beta = \frac{1}{it}$. *Factor* is an arbitrary constant since it depends on μ (see eq.(8)). We fix it in such a way that equation (9) holds. The implications of our formula (9) for the character expansion on $U(m|n)$, together with a detailed calculation of the correlators of superunitary matrices will be discussed elsewhere. Here we content ourselves with the following result

$$\int [dU] [\text{Str}(\Lambda U \Omega U^\dagger)]^k = \begin{cases} 0 & \text{if } k = 0, \dots, mn - 1 \\ (mn)! \Sigma(\lambda, \bar{\lambda}) \Sigma(\omega, \bar{\omega}) & \text{if } k = mn. \end{cases} \quad (39)$$

The simplest identities that follow from the above result are

$$\int [dU] U_{i_1 j_1} U_{j_1 i_1}^\dagger \dots U_{i_r j_r} U_{j_r i_r}^\dagger = 0, \text{ for } r = 1, 2 \dots mn - 1, \quad (40)$$

$$\int [dU] U_{11} U_{11}^\dagger \dots U_{mm} U_{mm}^\dagger = \frac{1}{m!}, \text{ for } U \in U(m|1), \quad (41)$$

$$\int [dU] U_{22} U_{22}^\dagger \dots U_{1+n \ 1+n} U_{1+n \ 1+n}^\dagger = \frac{(-1)^n}{n!}, \text{ for } U \in U(1|n). \quad (42)$$

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